Math 53: Multivariable Calculus **Sections 201, 204**

Worksheet for 2021-10-25

Conceptual questions

Question 1. Let $\mathbf{r}(t) = \langle f(t), g(t) \rangle$ be a parametrization of a curve C in $\mathbb{R}^2$. Explain how the "infinitesimal quantities"

$$\mathrm{d}x,\ \mathrm{d}y,\ \mathrm{d}s$$

are related to $\mathrm{d}t$, where s as usual denotes arclength measured along C.

Question 2. Let R be the filled-in ellipse $4x^2 + 9y^2 \leq 36$. If given an integral such as

$$\iint_R \sin(4x^2 + 9y^2)\,\mathrm{d}x\,\mathrm{d}y,$$

a natural change of variables to apply is $x = 3u$, $y = 2v$, because it converts the region to the circle $u^2 + v^2 \leq 1$. After you do that though, you'll want to apply another change of variables $u = r\cos\theta$, $v = r\sin\theta$.

Alternatively, you could do it in a single step by using the change of variables $x = 3r\cos\theta$, $y = 2r\sin\theta$. Compare these two approaches (fully evaluating the integral is unnecessary).

The following two questions are very tangential to the course content, so feel free to skip ahead to the computations.

Question 3. Let α be a fixed angle. Compute the Jacobian determinant of the (linear) transformation

$$x = (\cos\alpha)u - (\sin\alpha)v$$
$$y = (\sin\alpha)u + (\cos\alpha)v$$

Do you recognize what this transformation is? Hint: if $u = r\cos\theta$ and $v = r\sin\theta$, try computing x, y in terms of r, θ (you will need to recall some trigonometric identities). Explain the value of the Jacobian determinant geometrically.

Question 4. When performing change of variables on an integral, it is important to take the absolute value of the Jacobian determinant, which itself can be negative. Geometrically, what does a negative Jacobian determinant mean? Here's a simple concrete example to think about: consider $x = u^2$, $y = v$. The Jacobian determinant is negative for $u < 0$ and positive for $u > 0$.

Computations

Problem 1. Let C be a circle of radius R centered at the origin $(0, 0)$. Evaluate the integral

$$\int_C \frac{-x}{x^2 + y^2}\,\mathrm{d}s.$$

(It turns out that this integral has the same value even if C is not centered at $(0, 0)$, as long as it encloses $(0, 0)$. This, however, is quite hard to show.)

Problem 2. Let C be the portion of the curve $y = x^2$ starting at $(0, 0)$ and ending at $(2, 4)$. Evaluate the integral

$$\int_C x\,\mathrm{d}s.$$

If C started at $(2, 4)$ and ended at $(0, 0)$ instead, would your answer change?

Problem 3. Let C be the portion of the curve $y^2 = x^3 + x^2 + 1$ starting at $(0, -1)$ and ending at $(0, 1)$. Evaluate the integral

$$\int_C \frac{x^3 + x^2 + 2}{y^2 - x^3 - x^2}\,\mathrm{d}y.$$

If C started at $(0, 1)$ and ended at $(0, -1)$ instead, would your answer change?